

RESEARCH

Open Access

A supplement to the convergence rate in a theorem of Heyde

Jianjun He* and Tingfan Xie

*Correspondence: hejj@cjlw.edu.cn
Department of Mathematics, China
Jiliang University, Hangzhou,
310018, China**Abstract**

Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with zero mean, set $S_n = \sum_{k=1}^n X_k$, $EX^2 = \sigma^2 > 0$, and $\lambda(\epsilon) = \sum_{n=1}^{\infty} P(|S_n| \geq n\epsilon)$. In this paper, the authors discuss the rate of approximation of σ^2 by $\epsilon^2 \lambda(\epsilon)$ under suitable conditions, improve the results of Klesov (Theory Probab. Math. Stat. 49:83-87, 1994), and extend the work He and Xie (Acta Math. Appl. Sin. 2012, doi:10.1007/s10255-012-0138-6).

MSC: 60F15; 60G50**Keywords:** convergence rate; i.i.d. random variable; theorem of Heyde

1 Introduction and main results

Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables, set $S_n = \sum_{k=1}^n X_k$, and $\lambda(\epsilon) = \sum_{n=1}^{\infty} P(|S_n| \geq n\epsilon)$. Heyde [1] proved that

$$\lim_{\epsilon \rightarrow 0} \epsilon^2 \lambda(\epsilon) = \sigma^2,$$

whenever $EX^2 = \sigma^2 < \infty$ and $EX = 0$.

There are various extensions of this result: Chen [2], Gut and Spätara [3], Lanzinger and Stadtmüller [4]. Liu and Lin [5] introduced a new kind of complete moment convergence; Klesov [6] studied the rate of approximation of σ^2 by $\epsilon^2 \lambda(\epsilon)$ and proved the following Theorem A.

Theorem A *Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with zero mean, if $EX^2 = \sigma^2 > 0$, and $E|X|^3 < \infty$, then*

$$\epsilon^2 \lambda(\epsilon) - \sigma^2 = o(\epsilon^{1/2}), \quad \text{as } \epsilon \rightarrow 0.$$

Recently, He and Xie [7] obtained Theorem B which improved Theorem A. Gut and Steinebach [8] extended the results of Klesov [6].

Theorem B *Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables, and $0 < \delta \leq 1$, if*

$$EX = 0, \quad EX^2 = \sigma^2 > 0 \quad \text{and} \quad E|X|^{2+\delta} < \infty,$$

then

$$\epsilon^2 \lambda(\epsilon) - \sigma^2 = \begin{cases} O(\epsilon), & \delta = 1, \\ o(\epsilon^\delta), & 0 < \delta < 1. \end{cases}$$

Let G be the set of functions $g(x)$ that are defined for all real x and satisfy the following conditions: (a) $g(x)$ is nonnegative, even, nondecreasing in the interval $x > 0$, and $g(x) \neq 0$ for $x \neq 0$; (b) $\frac{x}{g(x)}$ is nondecreasing in the interval $x > 0$.

Let G_0 be the set of functions $g(x) \in G$ satisfying the supplementary condition (c) $\lim_{x \rightarrow \infty} \frac{g(x^2)}{xg(x)} = 0$. Obviously, the function $g(x) = |x|^\delta$ with $0 < \delta < 1$ belongs to G_0 and does not belong to G_0 if $\delta = 1$. The purpose of this paper is to generalize Theorem B to the case where the condition $E|X|^{2+\delta} < \infty$ is replaced by a more general condition $E|X|^2 g(X) < \infty$ in which the function g belongs to some subset of G . Denote $T_g(v) = EX^2 g(X) I(|X| > v)$, $T_g(v)$ is a nonnegative nonincreasing function in the interval $v > 0$, and $\lim_{v \rightarrow \infty} T_g(v) = 0$ with $EX^2 g(X) < \infty$. Now we state our results as follows.

Theorem 1.1 *Let $\{X, X_n; n \geq 1\}$ be a sequence of i.i.d. random variables with zero mean and $EX^2 = \sigma^2 > 0$, if $EX^2 g(X) < \infty$ for some function $g(x) \in G$, and*

$$\sum_{n=1}^{\infty} \frac{1}{ng(\sqrt{n})} < \infty, \quad (1.1)$$

then

$$\epsilon^2 \lambda(\epsilon) - \sigma^2 = O(\epsilon^{1/2}) + o(1)(h_1(\epsilon) + f_1(\epsilon)), \quad \text{as } \epsilon \rightarrow 0, \quad (1.2)$$

$$\text{where } f_1(\epsilon) = \sum_{n=\lfloor \frac{1}{\epsilon^2} \rfloor + 1}^{\infty} \frac{1}{ng(\sqrt{n})}, \quad h_1(\epsilon) = \epsilon^2 \sum_{n=1}^{\lfloor \frac{1}{\epsilon^2} \rfloor} \frac{1}{g(\sqrt{n})}.$$

Theorem 1.2 *Under the conditions of Theorem 1.1, and $g(x) \in G_0$, then*

$$\epsilon^2 \lambda(\epsilon) - \sigma^2 = o(1)(h_1(\epsilon) + f_1(\epsilon)), \quad \text{as } \epsilon \rightarrow 0. \quad (1.3)$$

Throughout this paper, we suppose that C denotes a constant which only depends on some given numbers and may be different at each appearance, and that $[x]$ denotes the integer part of x .

2 Proofs of the main results

Before we prove the main results we state some lemmas. Lemma 2.1 is from [7]. $\Phi(x)$ is the standard normal distribution function, $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt$.

Lemma 2.1 *Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. standard normal distribution random variables. Then*

$$\epsilon^2 \lambda(\epsilon) = \epsilon^2 \sum_{n=1}^{\infty} \frac{2}{\sqrt{2\pi}} \int_{\epsilon\sqrt{n}}^{\infty} e^{-t^2/2} dt = 1 - \frac{\epsilon^2}{2} + O(\epsilon^3), \quad \text{as } \epsilon \rightarrow 0. \quad (2.1)$$

If $\{X_n, n \geq 1\}$ is a sequence of independent random variables with zero mean and finite variance, and put $EX_j^2 = \sigma_j^2$, $B_n = \sum_{j=1}^n \sigma_j^2$, Bikelis [9] obtained the following inequality:

$$\begin{aligned} & \left| P\left(\frac{1}{\sqrt{B_n}} \sum_{j=1}^n X_j < x\right) - \Phi(x) \right| \\ & \leq C \left\{ B_n^{-1} (1 + |x|)^{-2} \sum_{j=1}^n \int_{|u| > (1+|x|)B_n^{1/2}} u^2 dV_j(u) \right. \\ & \quad \left. + B_n^{-3/2} (1 + |x|)^{-3} \sum_{j=1}^n \int_{|u| \leq (1+|x|)B_n^{1/2}} |u|^3 dV_j(u) \right\}, \end{aligned}$$

for every x , where $V_j(x) = P(X_j < x)$ is the distribution function of the random variable X_j . By applying the above inequality to the sequence of i.i.d. random variables with zero mean and variance 1, and letting $|x| = \epsilon\sqrt{n}$, we have the following lemma.

Lemma 2.2 Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with zero mean and $EX^2 = 1$. Then for any given $\epsilon > 0$, we have

$$\begin{aligned} & \left| P(|S_n| > n\epsilon) - \frac{2}{\sqrt{2\pi}} \int_{\epsilon\sqrt{n}}^{\infty} e^{-t^2/2} dt \right| \\ & \leq C(1 + \epsilon\sqrt{n})^{-2} \int_{|u| > (1+\epsilon\sqrt{n})\sqrt{n}} u^2 dV(u) \\ & \quad + Cn^{-1/2}(1 + \epsilon\sqrt{n})^{-3} \int_{|u| \leq (1+\epsilon\sqrt{n})\sqrt{n}} |u|^3 dV(u), \end{aligned}$$

where $V(x) = P(X < x)$ is the distribution function of a random variable X .

Proof of Theorem 1.1 Without loss of generality, we suppose that $\sigma^2 = 1$, $0 < \epsilon < 1$, and write

$$\epsilon^2 \lambda(\epsilon) = I + \epsilon^2 \sum_{n=1}^{\infty} \frac{2}{\sqrt{2\pi}} \int_{\epsilon\sqrt{n}}^{\infty} e^{-t^2/2} dt,$$

where

$$I = \epsilon^2 \sum_{n=1}^{\infty} \left(P(|S_n| > n\epsilon) - \frac{2}{\sqrt{2\pi}} \int_{\epsilon\sqrt{n}}^{\infty} e^{-t^2/2} dt \right).$$

Applying Lemma 2.1, we obtain

$$\epsilon^2 \lambda(\epsilon) = I + 1 - \frac{\epsilon^2}{2} + O(\epsilon^3),$$

then

$$\epsilon^2 \lambda(\epsilon) - 1 = -\frac{\epsilon^2}{2} + \epsilon^2 \sum_{n=1}^{\infty} R_n + O(\epsilon^3),$$

here $R_n = P(|S_n| > n\epsilon) - \frac{2}{\sqrt{2\pi}} \int_{\epsilon\sqrt{n}}^{\infty} e^{-t^2/2} dt$. By Lemma 2.2,

$$|R_n| \leq R_{1n} + R_{2n},$$

where

$$R_{1n} = C(1 + \epsilon\sqrt{n})^{-2} \int_{|u| > (1 + \epsilon\sqrt{n})\sqrt{n}} u^2 dV(u),$$

$$R_{2n} = Cn^{-1/2}(1 + \epsilon\sqrt{n})^{-3} \int_{|u| \leq (1 + \epsilon\sqrt{n})\sqrt{n}} |u|^3 dV(u).$$

We obtain

$$\epsilon^2 \lambda(\epsilon) - 1 = \epsilon^2 \sum_{n=1}^{\infty} R_{1n} + \epsilon^2 \sum_{n=1}^{\infty} R_{2n} + O(\epsilon^2). \quad (2.2)$$

Firstly, we estimate $\epsilon^2 \sum_{n=1}^{\infty} R_{1n}$. Note that

$$\epsilon^2 \sum_{n=1}^{\infty} R_{1n} = \epsilon^2 \sum_{n=1}^{\lfloor \frac{1}{\epsilon^2} \rfloor} R_{1n} + \epsilon^2 \sum_{n=\lfloor \frac{1}{\epsilon^2} \rfloor + 1}^{\infty} R_{1n} =: T_1 + T_2.$$

Applying the condition $EX^2g(X) < \infty$, we have

$$\lim_{n \rightarrow \infty} \int_{|u| > \sqrt[4]{n}} u^2 g(u) dV(u) = 0.$$

Therefore, for any $\eta > 0$, there is an integer N_0 such that $\int_{|u| > \sqrt[4]{n}} u^2 g(u) dV(u) \leq \eta$, whenever $n > N_0$. Hence

$$\begin{aligned} T_1 &\leq C\epsilon^2 \sum_{n=1}^{N_0} \int_{|u| > \sqrt{n}} u^2 dV(u) + C\epsilon^2 \sum_{n=N_0+1}^{\lfloor \frac{1}{\epsilon^2} \rfloor} (1 + \epsilon\sqrt{n})^{-2} \int_{|u| > (1 + \epsilon\sqrt{n})\sqrt{n}} u^2 dV(u) \\ &\leq C\epsilon^2 N_0 + C\epsilon^2 \eta \sum_{n=N_0+1}^{\lfloor \frac{1}{\epsilon^2} \rfloor} \frac{1}{(1 + \epsilon\sqrt{n})^2 g(\sqrt{n}(1 + \epsilon\sqrt{n}))} \\ &\leq C\epsilon^2 \left(N_0 + \eta \sum_{n=1}^{\lfloor \frac{1}{\epsilon^2} \rfloor} \frac{1}{g(\sqrt{n})} \right) \\ &= Ch_1(\epsilon) \left(\frac{N_0}{\sum_{n=1}^{\lfloor \frac{1}{\epsilon^2} \rfloor} \frac{1}{g(\sqrt{n})}} + \eta \right) \\ &\leq Ch_1(\epsilon)(N_0\epsilon + \eta) \\ &= o(h_1(\epsilon)), \end{aligned} \quad (2.3)$$

where $h_1(\epsilon) = \epsilon^2 \sum_{n=1}^{\lfloor \frac{1}{\epsilon^2} \rfloor} \frac{1}{g(\sqrt{n})}$. For T_2 , noting that $g(x) \in G$, we have the following inequality:

$$\begin{aligned} T_2 &\leq C\epsilon^2 \sum_{n=\lfloor \frac{1}{\epsilon^2} \rfloor+1}^{\infty} \frac{1}{n\epsilon^2} \int_{|u|>\sqrt{n}(1+\epsilon\sqrt{n})} u^2 dV(u) \\ &\leq C \sum_{n=\lfloor \frac{1}{\epsilon^2} \rfloor+1}^{\infty} \frac{1}{ng(\sqrt{n}(1+\epsilon\sqrt{n}))} \int_{|u|>\sqrt{n}(1+\epsilon\sqrt{n})} u^2 g(u) dV(u) \\ &\leq C \sum_{n=\lfloor \frac{1}{\epsilon^2} \rfloor+1}^{\infty} \frac{1}{ng(\sqrt{n})} \int_{|u|>\frac{1}{\epsilon}} u^2 g(u) dV(u) \\ &\leq CT_g\left(\frac{1}{\epsilon}\right)f_1(\epsilon). \end{aligned} \quad (2.4)$$

Next, we estimate the second term of (2.2). Note that

$$\begin{aligned} \epsilon^2 \sum_{n=1}^{\infty} R_{2n} &= C\epsilon^2 \sum_{n=1}^{\infty} n^{-1/2}(1+\epsilon\sqrt{n})^{-3} \int_{|u|\leq(\sqrt{n}(1+\epsilon\sqrt{n}))^{1/2}} |u|^3 dV(u) \\ &\quad + C\epsilon^2 \sum_{n=1}^{\infty} n^{-1/2}(1+\epsilon\sqrt{n})^{-3} \int_{(\sqrt{n}(1+\epsilon\sqrt{n}))^{1/2}<|u|<\sqrt{n}(1+\epsilon\sqrt{n})} |u|^3 dV(u) \\ &=: J_1 + J_2. \end{aligned}$$

For J_1 , we can write

$$\begin{aligned} J_1 &= C\epsilon^2 \left(\sum_{n=1}^{\lfloor \frac{1}{\epsilon^2} \rfloor} + \sum_{n=\lfloor \frac{1}{\epsilon^2} \rfloor+1}^{\infty} \right) n^{-1/2}(1+\epsilon\sqrt{n})^{-3} \int_{|u|\leq(\sqrt{n}(1+\epsilon\sqrt{n}))^{1/2}} |u|^3 dV(u) \\ &=: J_{11} + J_{12}. \end{aligned}$$

Noting that $\frac{x}{g(x)}$ is nondecreasing in the interval $x > 0$, we have

$$\begin{aligned} J_{11} &= C\epsilon^2 \sum_{n=1}^{\lfloor \frac{1}{\epsilon^2} \rfloor} \frac{1}{\sqrt{n}(1+\epsilon\sqrt{n})^3} \int_{|u|\leq(\sqrt{n}(1+\epsilon\sqrt{n}))^{1/2}} |u|^3 dV(u) \\ &\leq C\epsilon^2 \sum_{n=1}^{\lfloor \frac{1}{\epsilon^2} \rfloor} \frac{1}{n^{1/4}(1+\epsilon\sqrt{n})^{5/2}g((\sqrt{n}(1+\epsilon\sqrt{n}))^{1/2})} \int_{|u|\leq(\sqrt{n}(1+\epsilon\sqrt{n}))^{1/2}} u^2 g(u) dV(u) \\ &\leq C\epsilon^2 \sum_{n=1}^{\lfloor \frac{1}{\epsilon^2} \rfloor} \frac{1}{n^{1/4}g(n^{1/4})} \\ &= Ch_2(\epsilon), \end{aligned} \quad (2.5)$$

where $h_2(\epsilon) = \epsilon^2 \sum_{n=1}^{\lfloor \frac{1}{\epsilon^2} \rfloor} \frac{1}{n^{1/4}g(n^{1/4})}$.

Similarly, we can obtain

$$\begin{aligned}
 J_{12} &= C\epsilon^2 \sum_{n=[\frac{1}{\epsilon^2}]+1}^{\infty} \frac{1}{\sqrt{n}(1+\epsilon\sqrt{n})^3} \int_{|u| \leq (\sqrt{n}(1+\epsilon\sqrt{n}))^{1/2}} |u|^3 dV(u) \\
 &\leq C\epsilon^2 \sum_{n=[\frac{1}{\epsilon^2}]+1}^{\infty} \frac{1}{n^{1/4}(1+\epsilon\sqrt{n})^{5/2}g((\sqrt{n}(1+\epsilon\sqrt{n}))^{1/2})} \int_{|u| \leq (\sqrt{n}(1+\epsilon\sqrt{n}))^{1/2}} u^2 g(u) dV(u) \\
 &\leq C\epsilon^2 \sum_{n=[\frac{1}{\epsilon^2}]+1}^{\infty} \frac{1}{\epsilon^{5/2}n^{3/2}g(n^{1/4})} \\
 &= C \frac{1}{\sqrt{\epsilon}} f_2(\epsilon),
 \end{aligned} \tag{2.6}$$

where $f_2(\epsilon) = \sum_{n=[\frac{1}{\epsilon^2}]+1}^{\infty} \frac{1}{n^{3/2}g(n^{1/4})}$.

For J_2 , we write

$$\begin{aligned}
 J_2 &= C\epsilon^2 \left(\sum_{n=1}^{[\frac{1}{\epsilon^2}]} + \sum_{n=[\frac{1}{\epsilon^2}]+1}^{\infty} \right) n^{-1/2} (1+\epsilon\sqrt{n})^{-3} \int_{(\sqrt{n}(1+\epsilon\sqrt{n}))^{1/2} < |u| < \sqrt{n}(1+\epsilon\sqrt{n})} |u|^3 dV(u) \\
 &=: J_{21} + J_{22}.
 \end{aligned}$$

Using the properties of $g(x)$ by simple calculation, it follows that

$$\begin{aligned}
 J_{21} &= C\epsilon^2 \sum_{n=1}^{[\frac{1}{\epsilon^2}]} n^{-1/2} (1+\epsilon\sqrt{n})^{-3} \int_{(\sqrt{n}(1+\epsilon\sqrt{n}))^{1/2} < |u| < \sqrt{n}(1+\epsilon\sqrt{n})} |u|^3 dV(u) \\
 &\leq C\epsilon^2 \left(\sum_{n=1}^{N_0} + \sum_{n=N_0+1}^{[\frac{1}{\epsilon^2}]} \right) \frac{1}{(1+\epsilon\sqrt{n})^2 g(\sqrt{n}(1+\epsilon\sqrt{n}))} \\
 &\quad \times \int_{(\sqrt{n}(1+\epsilon\sqrt{n}))^{1/2} < |u| < \sqrt{n}(1+\epsilon\sqrt{n})} u^2 g(u) dV(u) \\
 &\leq C\epsilon^2 \left(\sum_{n=1}^{N_0} + \sum_{n=N_0+1}^{[\frac{1}{\epsilon^2}]} \right) \frac{1}{g(\sqrt{n})} \int_{|u| > n^{1/4}} u^2 g(u) dV(u) \\
 &\leq C\epsilon^2 \left(N_0 + \eta \sum_{n=1}^{[\frac{1}{\epsilon^2}]} \frac{1}{g(\sqrt{n})} \right) \\
 &= o(h_1(\epsilon)),
 \end{aligned} \tag{2.7}$$

and

$$\begin{aligned}
 J_{22} &\leq C\epsilon^2 \sum_{n=[\frac{1}{\epsilon^2}]+1}^{\infty} n^{-\frac{1}{2}} (1+\epsilon\sqrt{n})^{-3} \int_{(\sqrt{n}(1+\epsilon\sqrt{n}))^{1/2} < |u| < \sqrt{n}(1+\epsilon\sqrt{n})} |u|^3 dV(u) \\
 &\leq C \sum_{n=[\frac{1}{\epsilon^2}]+1}^{\infty} \frac{1}{ng(\sqrt{n})} \int_{(\sqrt{n}(1+\epsilon\sqrt{n}))^{1/2} < |u| < \sqrt{n}(1+\epsilon\sqrt{n})} u^2 g(u) dV(u)
 \end{aligned}$$

$$\begin{aligned} &\leq CT_g\left(\frac{1}{\sqrt{\epsilon}}\right) \sum_{n=[\frac{1}{\epsilon^2}]+1}^{\infty} \frac{1}{ng(\sqrt{n})} \\ &\leq CT_g\left(\frac{1}{\sqrt{\epsilon}}\right) f_1(\epsilon). \end{aligned} \quad (2.8)$$

From (2.2) to (2.8), we conclude that

$$\epsilon^2 \lambda(\epsilon) - 1 \leq C \frac{1}{\sqrt{\epsilon}} f_2(\epsilon) + CT_g\left(\frac{1}{\sqrt{\epsilon}}\right) f_1(\epsilon) + o(1)h_1(\epsilon) + Ch_2(\epsilon). \quad (2.9)$$

Since

$$\frac{1}{\sqrt{\epsilon}} f_2(\epsilon) \leq \frac{C}{\sqrt{\epsilon}} \sum_{n=[\frac{1}{\epsilon^2}]+1}^{\infty} \frac{1}{n^{3/2}} \leq C\sqrt{\epsilon},$$

and

$$h_2(\epsilon) = \epsilon^2 \sum_{n=1}^{[\frac{1}{\epsilon^2}]} \frac{1}{\sqrt[4]{n}g(\sqrt[4]{n})} \leq C\epsilon^2 \sum_{n=1}^{[\frac{1}{\epsilon^2}]} \frac{1}{\sqrt[4]{n}} \leq C\sqrt{\epsilon},$$

by (2.9), we have

$$\epsilon^2 \lambda(\epsilon) - 1 = O(\epsilon^{1/2}) + o(1)(f_1(\epsilon) + h_1(\epsilon)).$$

This completes the proof of Theorem 1.1. \square

Proof of Theorem 1.2 By the conditions $g(x) \in G_0$, and $\lim_{x \rightarrow \infty} \frac{g(x^2)}{xg(x)} = 0$, for any $\eta > 0$, there is an integer N_1 such that $\frac{g(\sqrt{n})}{\sqrt[4]{n}g(\sqrt[4]{n})} \leq \eta$, whenever $n > N_1$. We have

$$\begin{aligned} h_2(\epsilon) &\leq \epsilon^2 \sum_{n=1}^{N_1} \frac{1}{\sqrt[4]{n}g(\sqrt[4]{n})} + \epsilon^2 \sum_{n=N_1+1}^{[\frac{1}{\epsilon^2}]} \frac{\eta}{g(\sqrt{n})} \\ &\leq C\epsilon^2 N_1 + \epsilon^2 \sum_{n=N_1+1}^{[\frac{1}{\epsilon^2}]} \frac{\eta}{g(\sqrt{n})} \\ &\leq C\epsilon^2 N_1 + \epsilon^2 \sum_{n=1}^{[\frac{1}{\epsilon^2}]} \frac{\eta}{g(\sqrt{n})} \\ &= o(1)h_1(\epsilon), \end{aligned} \quad (2.10)$$

and

$$\begin{aligned} \frac{1}{\sqrt{\epsilon}} f_2(\epsilon) &\leq \frac{1}{\sqrt{\epsilon}} \sum_{n=[\frac{1}{\epsilon^2}]+1}^{\infty} \frac{\eta}{n^{5/4}g(\sqrt{n})} \\ &\leq \sum_{n=[\frac{1}{\epsilon^2}]+1}^{\infty} \frac{\eta}{ng(\sqrt{n})} = o(1)f_1(\epsilon). \end{aligned} \quad (2.11)$$

By (2.9)-(2.11), note that $T_g(\frac{1}{\sqrt{\epsilon}}) = o(1)$, as $\epsilon \rightarrow 0$, we have

$$\epsilon^2 \lambda(\epsilon) - \sigma^2 = o(1)(h_1(\epsilon) + f_1(\epsilon)), \quad \text{as } \epsilon \rightarrow 0.$$

This completes the proof of Theorem 1.2. \square

Remark 2.1 If $g(x) = |x|^\delta$, $0 < \delta < 1$, then $f_1(\epsilon) = O(\epsilon^\delta)$, $h_1(\epsilon) = O(\epsilon^\delta)$. By Theorem 1.2, we get

$$\epsilon^2 \lambda(\epsilon) - \sigma^2 = o(\epsilon^\delta), \quad \text{as } \epsilon \rightarrow 0.$$

Remark 2.2 If $g(x) = |x|$, $\delta = 1$, then $\frac{1}{\sqrt{\epsilon}} f_2(\epsilon) = O(\epsilon)$, $f_1(\epsilon) = O(\epsilon)$, $h_1(\epsilon) = O(\epsilon)$, $h_2(\epsilon) = O(\epsilon)$. By (2.9), we get

$$\epsilon^2 \lambda(\epsilon) - \sigma^2 = O(\epsilon), \quad \text{as } \epsilon \rightarrow 0.$$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors read and approved the final manuscript.

Acknowledgements

The authors are very grateful to the referees and editors for their valuable comments and some helpful suggestions that improved the clarity and readability of the paper.

Received: 27 May 2012 Accepted: 20 August 2012 Published: 4 September 2012

References

- Heyde, CC: A supplement to the strong law of large numbers. *J. Appl. Probab.* **12**, 903-907 (1975)
- Chen, R: A remark on the strong law of large numbers. *Proc. Am. Math. Soc.* **61**, 112-116 (1976)
- Gut, A, Spătaru, A: Precise asymptotics in the Baum-Kate and Davis law of large numbers. *J. Math. Anal. Appl.* **248**, 233-246 (2000)
- Lanzinger, H, Stadtmüller, U: Refined Baum-Katz laws for weighted sums of iid random variables. *Stat. Probab. Lett.* **69**, 357-368 (2004)
- Liu, WD, Lin, ZY: Precise asymptotic for a new kind of complete moment convergence. *Stat. Probab. Lett.* **76**, 1787-1799 (2006)
- Klesov, OI: On the convergence rate in a theorem of Heyde. *Theory Probab. Math. Stat.* **49**, 83-87 (1994)
- He, JJ, Xie, TF: Asymptotic property for some series of probability. *Acta Math. Appl. Sin.* (2012). doi:10.1007/s10255-012-0138-6
- Gut, A, Steinebach, J: Convergence rates in Precise asymptotics. *J. Math. Anal. Appl.* **390**, 1-14 (2012)
- Bikelis, A: Estimates of the remainder in the central limit theorem. *Litovsk. Mat. Sb.* **6**, 323-346 (1966)

doi:10.1186/1029-242X-2012-195

Cite this article as: He and Xie: A supplement to the convergence rate in a theorem of Heyde. *Journal of Inequalities and Applications* 2012 **2012**:195.